VARIATIONAL PRINCIPLES FOR COMPRESSIBLE VISCOUS FLUID IN MAGNETOHYDRODYNAMICS

N. K. SHARMA* AND S. R. SHARMA

DEPARTMENT OF PHYSICS, UNIVERSITY OF RAJASTHAN, JAIPUR

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ABSTRACT. Variational principles with proper Lagrangian densities for the case of a magnetohydrodynamic viscous compressible fluid have been used to derive momentum equations and the equations governing the magnetic field. As an illustration the case of a perfectly conducting plasma continuum has been treated.

INTRODUCTION

Recently several papers (Herivel 1954, Katz 1961, Su 1961) have treated the variational principles for non-dissipative plasmas. Rosen (1958) has used a variational method for such a dissipative system using Onsagar's (1931) dissipation function ϕ. In this communication we have used a very straight forward and simple variational approach for deriving the momentum equations and the equations governing the magnetic field for a dissipative magnetohydrodynamic compressible fluid.

The actual formulation has been divided into two parts. In the first part we derive the momentum equations for the motion of the fluid, and the equations governing the magnetic field have been taken as admissibility conditions. In the second part the later have been derived treating the former as admissibility conditions. The method used in this formulation is due to Bateman (1931).

As an illustration we have considered the case of a perfectly conducting plasma continuum, our formulation gives exactly the equations for this continuum, which is really very straight forward and simple as compared to the recent derivation given by Lundgren (1963).

VARIATIONAL PRINCIPLE—I

The appropriate equations for the problem under consideration are (Chandrasekhar 1961)

$$\rho Du_i = X_i - \frac{\partial \varpi}{\partial x_i} + \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\} - \frac{2}{3}\frac{\partial}{\partial x_i}\left(\mu \frac{\partial u_k}{\partial x_k}\right) + H_j \frac{\partial H_i}{\partial x_j} \quad (1)$$

*Present address :—Tata Institute of Fundamental Research, Bombay 5, India.

with
$$\varpi = p + \frac{|\vec{H}|^2}{8\pi} \quad \cdots \quad (2)$$

The equation of motion governing the magnetic field is

$$\frac{\partial H_i}{\partial t} + \frac{\partial}{\partial x_j}(u_j H_i - u_i H_j) = \eta \nabla^2 H_i \quad \cdots \quad (3)$$

The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0. \quad \cdots \quad (4)$$

and the magnetic field satisfies the divergence condition,

$$\frac{\partial H_i}{\partial x_i} = 0 \quad \cdots \quad (5)$$

The symbols in the above equations have the following meanings :

ρ, denotes the density of the fluid and is a function of space coordinates i.e. $\rho = \rho(x, y, z)$

U_i, denotes the i-th component of the velocity vector.

X_i, denotes the i-th component of the external forces of non-magnetic origin.

μ, is the viscosity of the fluid assumed to be a function of space coordinates i.e. $\mu = \mu(x, y, z)$.

p, denotes the fluid pressure.

H_i is the i-th component of magnetic intensity.

η, denotes the resistivity, assumed constant.

ϖ, is a function involving p and $\vec{H}$ as defined in eq (2) and D is the total derivative.

It may be pointed out that throughout the formulations the method of Cartesian tensors has been used. In this section the appropriate Euler-Lagrangian density yield equations (1) and equation (3) has been taken as admissibility condition. The boundary conditions are the usual ones, namely, the fluid under consideration is supposed to occupy a finite region of space whose boundary is formed partly of the surfaces with motions and partly of free surfaces. The time interval of the motion of the fluid is considered from $t = t_0$ to $t = t_1$, and the region occupied by the fluid at these instants is R_0 and R_1 respectively with the corresponding boundaries as S_0 and S_1 respectively. x_i, Dx_i are continuous and δx_i is zero at $t = t_0$ and $t = t_1$.

For simplicity we define the following symbols :

$$a_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \ldots (6)$$

$$b_{ij} = \frac{1}{2}\left(\frac{\partial H_i}{\partial x_j} + \frac{\partial H_j}{\partial x_i}\right) \quad \ldots (7)$$

$$C_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \quad \ldots (8)$$

and

$$u_i = Dx_i \quad \ldots (9)$$

We have taken $U_i = \delta xi$, therefore $DU_i = D\delta x_i = \delta Dx_i = \delta u_i$ ... (10)

We define the Ligrangian density L as follows :

$$\delta \int_{t_0}^{t_1} L\,dt = \delta \int_{t_0}^{t_1} (A+B+C)dt = 0. \quad \ldots (11)$$

Where A, B and C have been defined as the volume integrals as follows.

$$\delta A = \int_R U_i D\rho\, \delta x_i d\tau + \int_R \rho U_i \delta U_i d\tau$$

$$= \int_R U_i D\rho \delta x_i d\tau + \int_R \rho U_i D\delta x_i d\tau$$

$$= \int_R U_i D(\rho \delta x_i) d\tau \quad \ldots (12)$$

Here $d\tau$ denotes the volume element $dxdydz$ of the fluid under consideration. Therefore,

$$\int_{t_0}^{t_1} \delta A dt = \int_{t_0}^{t_1} dt \int_R U_i D(\rho \delta x_i) d\tau$$

$$= \int_R d\tau [U_i \rho \delta x_i]_{t_0}^{t_1} - \int_{t_0}^{t_1} dt \int_R DU_i \rho \delta x_i d\tau$$

$$= -\int_{t_0}^{t_1} dt \int_R \rho\, DU_i \delta x_i d\tau \quad \ldots (13)$$

Next,

$$\delta B = -2\int_R \mu C_{ij} a_{ij} d\tau - \int_R H_i H_j \frac{\partial U_i}{\partial x_j} d\tau + \frac{2}{3}\int_R \mu a_{kk} \frac{\partial U_i}{\partial x_i} d\tau$$

$$= -\int_R [\mu a_{ij}(\delta x_i)_{,j} + (\mu a_{ij})(\delta x_j)_{,i}] - \int_R H_i H_j (\delta x_i)_{,j} d\tau + \frac{2}{3}\int_R \mu a_{kk}(\delta x_i)_{,i} d\tau$$

$$\delta B = -\int_R [(\mu a_{ij}\delta x_i)_{,j} - (\mu a_{ij})_{,j}\delta x_i] d\tau$$

$$- \int_R [(\mu a_{ij}\delta x_j)_{,i} - (\mu a_{ij})_{,i}\delta x_j] d\tau$$

$$- \int_R [(H_i H_j \delta x_i)_{,j} - (H_i H_j)_{,j}\delta x_i]\, d\tau$$

$$+ \frac{2}{3}\int_R [(\mu a_{kk}\delta x_i)_{,i} - (\mu a_{kk})_{,i}\delta x_i] d\tau \qquad \ldots (14)$$

The application of Gauss' theorem and a little simplification yields

$$\delta B = -2\int_S \mu a_{ij} l_j \delta x_i ds + 2\int_R \frac{\partial}{\partial x_j}(\mu a_{ij})\delta x_i d\tau$$

$$- \int_S H_i H_j l_j \delta x_i ds + \int_R \frac{\partial}{\partial x_j}(H_i H_j)\delta x_i d\tau$$

$$+ \frac{2}{3}\int_S \mu a_{kk} l_i \delta x_i ds - \frac{2}{3}\int_R \frac{\partial}{\partial x_i}(\mu a_{kk})\delta x_i d\tau \qquad \ldots (15)$$

Lastly we define

$$\delta c = \int_R \left(X_i - \frac{\partial \varpi}{\partial x_i}\right)\delta x_i d\tau + \int_S X_{0i}\delta x_i ds \qquad \ldots (16)$$

Where X_{0i} are the components of surface traction. Therefore, the variational principle is

$$\delta\int_{t_0}^{t_1}(A+B+C)dt = \int_{t_0}^{t_1} dt \int_R d\tau \Big[X_i - \frac{\partial \varpi}{\partial x_i} - \rho D u_i$$

$$+ 2\frac{\partial}{\partial x_j}(\mu a_{ij}) + \frac{\partial}{\partial x_j}(H_i H_j) - \frac{2}{3}\frac{\partial}{\partial x_i}(\mu a_{kk})\Big]\delta x_i$$

$$+ \int_{t_0}^{t_1} dt \int_S ds \Big[X_{0i} - H_i H_j l_j - 2\mu a_{ij} l_j + \frac{2}{3}\mu a_{kk} l_i \Big] = 0. \qquad \ldots (17)$$

This on further simplification gives

$$\int_{t_0}^{t_1} dt \int_R \delta x_i \left\{ X_i - \frac{\partial \varpi}{\partial x_i} - \rho D u_i + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \right.$$

$$\left. - \frac{2}{3} \frac{\partial}{\partial x_j} \left(\mu \frac{\partial u_k}{\partial x_k} \right) + H_j \frac{\partial H_i}{\partial x_j} \right\} d\tau$$

$$+ \int_{t_0}^{t_1} dt \int_S \left[X_{oi} - H_i H_j l_j - 2\mu a_{ij} l_j + \frac{2}{3} \mu a_{kk} l_i \right] ds = 0 \qquad \ldots (18)$$

Since the variations δx_i are arbitrary on S and also at each point interior to R, the above relations can be satisfied if

$$\rho D u_i = X_i - \frac{\partial \varpi}{\partial x_i} + \frac{\partial}{\partial x_j} \left\{ \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\}$$

$$- \frac{2}{3} \frac{\partial}{\partial x_i} \left(\mu \frac{\partial u_k}{\partial x_k} \right) + H_j \frac{\partial H_i}{\partial x_j} . \qquad \ldots (19)$$

under the natural boundary conditions

$$X_{oi} - H_i H_j l_j - 2\mu a_{ij} l_j + \frac{2}{3} \mu a_{kk} l_i = 0 \qquad \ldots (20)$$

Equation (16) are the hydromagnetic equations of motion (eq. 1) for the viscous, compressible fluid under natural boundary conditions specified by eq. (17).

VARIATIONAL PRINCIPLE—II

In this section we have taken equations of motion of the fluid as the admissibility conditions and the equation governing the motion of the magnetic field has been derived.

The appropriate Lagrangian for this problem is

$$\delta \int_{t_0}^{t_i} L dt = \delta \int_{t_0}^{t_1} (G + M + N) dt = 0 \qquad \ldots (21)$$

Where the quantities G, M and N are defined as follows :

The variation in G is given by

$$\delta G = \int_R H_i \delta u_i d\tau = \int_R H_i D(\delta x_i) d\tau \qquad \ldots (22)$$

Therefore,

$$\int_{t_0}^{t_i} \delta G dt = \int_{t_0}^{t_1} dt \int_R H_i D(\delta x_i) d\tau$$

$$= \int_R [H_i \delta x_i]_{t_0}^{t_1} d\tau - \int_{t_0}^{t_1} dt \int_R DH_i \delta x_i d\tau$$

$$= -\int_{t_0}^{t_1} dt \int_R DH_i \delta x_i d\tau \quad \ldots \quad (23)$$

Next the variation in M is defined by

$$\delta M = \int_S X_{0i} \delta x_i ds + \int_R u_j \frac{\partial H_i}{\partial x_j} \delta x_i d\tau$$

and therefore

$$\int_{t_0}^{t_1} \delta M dt = \int_{t_0}^{t_1} dt \int_S X_{0i} \delta x_i ds + \int_{t_0}^{t_1} dt \int_R u_j \frac{\partial H_i}{\partial x_j} \delta x_i d\tau \quad \ldots \quad (24)$$

Lastly we define

$$\delta N = -2 \int_R \eta C_{ij} b_{ij} d\tau - \int_R (u_i H_j - u_j H_i) \frac{\partial U_i}{\partial x_j} d\tau \quad \ldots \quad (25)$$

Proceeding as for eq. (14) and simplifying we have

$$\delta N = -\eta \int_R b_{ij}[(\delta x_i)_{,j} + (\delta x_j)_{,i}] d\tau - \int_R (u_i H_j - u_j H_i)(\delta x_i)_{,j} d\tau$$

$$= -2\eta \int_R b_{ij} (\delta x_i)_{,j} d\tau - \int_R (u_i H_j - u_j H_i)(\delta x_i)_{,j} d\tau$$

$$= -\int_R (2\eta b_{ij} l_j + u_i H_j l_j - u_j H_i l_j) \delta x_i ds$$

$$+ 2\eta \int_R \frac{\partial}{\partial x_j} (b_{ij}) \delta x_i d\tau + \int_R \frac{\partial}{\partial x_j} (u_i H_j - u_j H_i) \delta x_i d\tau \quad \ldots \quad (26)$$

Combining equations (26), (24) and (23) we have

$$\int_{t_0}^{t_1} dt \int_R \left\{ \eta \frac{\partial}{\partial x_j} \left(\frac{\partial H_i}{\partial x_j} + \frac{\partial H_j}{\partial x_i} \right) + \frac{\partial}{\partial x_j} (u_i H_j - u_j H_i) + u_j \frac{\partial H_i}{\partial x_j} - DH_i \right\} \delta x_i d\tau$$

$$- \int_{t_0}^{t_1} dt \int_S \{ 2 b_{ij} l_j \eta + u_i H_j l_j - u_j H_i l_j - X_{0i} \} \delta x_i = 0 \quad \ldots \quad (27)$$

But since the variations δx_i are arbitrary on S and also at each point interior to R, equation (27) can be satisfied if

$$\frac{\partial H_i}{\partial t}+\frac{\partial}{\partial x_j}(u_jH_i-u_iH_j)=\eta\nabla^2H_i \qquad \dots (28)$$

and

$$2\eta b_{ij}l_j+u_iH_jl_j-u_jH_il_j-X_{0i}=0 \qquad \dots (29)$$

Equation (28) is the equation (3) governing motion of the magnetic field under the natural boundary-conditions specified by equation (29).

CASE OF A PERFECTLY CONDUCTING PLASMA CONTINUUM

As an illustration we consider the case of a perfectly conducting inviscid plasma continuum with density ρ and pressure p obviously in this case $\mu=0$, $\eta=0$ and $X_i=0$.

Using equation (11) we have

$$\delta\int_{t_0}^{t_1}L dt=\delta\int_{t_0}^{t_1}(A+B+C)dt=0. \qquad \dots (30)$$

where equation (13) gives

$$\delta A=-\int\rho Du_i\,\delta x_i d\tau \qquad \dots (31)$$

Equation (15) gives

$$\delta B=-\int_S H_iH_jl_j\delta x_i ds+\int_R\frac{\partial}{\partial x_j}(H_iH_j)\delta x_i\delta\tau \qquad \dots (32)$$

and equation (16) gives

$$\delta C=\int_S X_{0i}\delta x_i ds-\int_R\left(\frac{\partial\varpi}{\partial x_i}\right)\delta x_i d\tau \qquad \dots (33)$$

where X_{0i} are the components of surface traction. On substituting in (30) we have

$$\delta\int_{t_0}^{t_1}L\,dt=\delta\int_{t_0}^{t_1}(A+B+C)dt$$

$$=\int_{t_0}^{t_1}dt\int d\tau\left[\frac{\partial}{\partial x_j}(H_iH_j)-\rho Du_i-\frac{\partial\varpi}{\partial x_i}\right]\delta x_i$$

$$+\int_S(X_{0i}-H_iH_jl_j)\delta x_i ds=0 \qquad \dots (34)$$

Since the variations δx_i on S and also at each point interior to R are arbitrary, therefore equation (34) can be satisfied if

$$\rho Du_i + \frac{\partial \varpi}{\partial x_i} - \frac{\partial}{\partial x_j}(H_i H_j) = 0 \qquad \dots \quad (35)$$

under the boundary conditions

$$X_{oi} - H_i H_j l_j = 0. \qquad \dots \quad (36)$$

Equation (35) on further simplication gives

$$\rho \frac{du}{dt} = -\nabla p + J \times H \qquad \dots \quad (37)$$

Where $J = \frac{1}{4\pi} \nabla \times H$ Equation (37) is the desired equation for the Perfectly Conducting Plasma Continuum.

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